Single Currency Bermudan Swaption Valuation

The underlying security of a single currency Bermudan swaption is an interest-rate swap, which is specified by respective payer and receiver legs. Each of the legs above can pay a fixed rate, Libor or CMS rate. The owner of the Bermudan swaption can choose to enter into the swap above at certain pre-defined exercise times; upon exercise, the owner

- must pay all payer-leg quantities that reset on or after the exercise time, and
- will receive all receiver-leg quantities that reset on or after the exercise time.

The pricing method is based on Jamshidian's Libor rate model (i.e., where Libor rates are modeled simultaneously under the spot Libor measure). Furthermore, we value a Bermudan swaption based on the Monte Carlo technique presented by Longstaff and Schwartz towards American style pricing.

Let $T_1, \ldots, T_N$, where $0 < T_1 < \ldots < T_N$, be common Libor reset points. Here we assume that all interest rate reset and Bermudan exercise times belong to the set of common reset points, $\{T\}$.

We consider an interest-rate swap consisting of respective receiver and payer legs. Here the payer leg is specified by

- reset points, $t_1, \ldots, t_M$, where $\{t\} \subseteq \{T\}$ and $t_1 < \ldots < t_M$,
- an amount, $R(t_{i+1} - t_i)$, payable at time t_{i+1}, for $i = 1, \ldots, M-1$.

Here R can be a fixed rate, a Libor or a CMS rate. In the case of an n-period Libor rate,

$$R = \frac{1}{\sum_k \Delta_{p+k-1}} \left(\frac{1}{P(T_p, T_{p+n})} - 1 \right)$$

where

- $t_i = T_p$,
- $\Delta_i = T_{i+1} - T_i$,
- $P(t,T)$ is the price at time t of a zero-coupon bond, which matures at T and has $1 face value.

In the case of a single period Libor rate, for example, $R = \frac{1}{\Delta_p} \left(\frac{1}{P(T_p, T_{p+1})} - 1 \right)$.

For an m-period CMS rate with frequency, f (where f is a whole number of consecutive common reset periods),

$$R = \frac{1 - P(T_p, T_{p+fm})}{\sum_k \left(\sum_l \Delta_{p+f(k-1)+l-1} \right) P(T_p, T_{p+fk})}$$

For example, if $f = 1$ (i.e., the CMS has reset times that correspond to consecutive common reset points), then $R = \frac{1 - P(T_p, T_{p+m})}{\sum_k \Delta_{p+k-1} P(T_p, T_{p+k})}$.

Let the receiver leg be similarly defined with respect to the reset points, $\tau_1, \ldots, \tau_l$.

We now consider exercise points, $t'_1, \ldots, t'_p$, such that $\{t'\} \subseteq \{T\}$. Here the Bermudan swaption can be exercised at any point belonging to $\{t'\}$. If the option is exercised at time t'_k, for some $k \in \{1, \ldots, p\}$, then

- the owner must pay all payer-leg quantities that set at points t_i such that $t_i \geq t'_k$, and
- the owner receives all receiver-leg quantities that set at points τ_i such that $\tau_i \geq t'_k$.

We model Libor rates under the spot Libor measure, which has numeraire process,

$$N_{T_i} = \prod_{j=1}^{i} \frac{1}{P(T_{j-1}, T_j)},$$

where $P(t,T)$ denotes the price at time t of a zero coupon bond with maturity of T (ref https://finpricing.com/lib/FiBondCoupon.html).

Let $[t]$ denote the integer, i, such that $T_{i-1} < t \leq T_i$. Under the spot Libor measure, we assume that, for $i = 1, \ldots, N-1$ and $0 \leq t \leq T_i$,

$$dL_i = L_i\, \vec{\sigma}(t, T_i) \bullet \left(\sum_{j=[t]}^{i} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \vec{\sigma}(t, T_j) dt + d\vec{W} \right), \qquad (3.1.1)$$

where

- $\vec{W} \in \Re^4$ is a vector of uncorrelated, standard Brownian motions,
- $\vec{\sigma} \in \Re^4$ is a time deterministic volatility vector, which we define below.

We consider a volatility vector of the form

$$\vec{\sigma}(t,T) = \vec{\psi}(T-t)\upsilon(t,T).$$

Here

$$\begin{aligned}
\psi_1(t) &= c, \\
\psi_2(t) &= \sqrt{\frac{1-c^2}{1+(t/a)^{\alpha}}}, \\
\psi_3(t) &= \sqrt{(1-c^2)\left(1-\frac{1}{1+(t/a)^{\alpha}}\right)\left(\frac{1}{1+(t/b)^{\beta}}\right)}, \\
\psi_4(t) &= \sqrt{1-[\psi_1(t)]^2-[\psi_2(t)]^2-[\psi_3(t)]^2}.
\end{aligned}$$

Furthermore

$$\upsilon(t,T) = e^{\sum_i \sum_j a_{ij}\Phi_i(x_1)\Phi_j(x_2)}$$

where

$$\begin{aligned}
x_1 &= -1+2e^{-(T-t)/\tau_1}, \\
x_2 &= -1+2e^{-T/\tau_2}.
\end{aligned}$$

Here

$$\Phi_i(x) = \cos\left(i\cos^{-1}x\right)$$

denotes a Chebyshev polynomial of the first kind.

In the above, the parameters

$$a, b, c, \alpha, \beta, \tau_1, \tau_2, \left(a_{ij}\right), \tag{3.1.2}$$

are determined from calibration.

Let t' be an exercise time. Suppose that the payer leg has n future reset times, $t_1,..., t_n$, such that $t_1,...,t_n \geq t'$. Furthermore assume that an interest rate quantity, p_i, for $i = 1,...,n-1$, sets at t_i and is paid at t_{i+1}. Similarly assume that the receiver leg has m future reset times, $\tau_1,..., \tau_m$, where $\tau_1,...,\tau_m \geq t'$. Furthermore assume that an interest rate quantity, r_i, for $i = 1,...,m-1$, sets at τ_i and is received at τ_{i+1}.

The payoff from a European option to enter into the swap at t' is then given by

$$\max\left(N_{t'} \sum_{i=1}^{m-1} E\left(\frac{r_i}{N_{\tau_{i+1}}} \middle| F_{t'} \right) - N_{t'} \sum_{i=1}^{n-1} E\left(\frac{p_i}{N_{t_{i+1}}} \middle| F_{t'} \right),\ 0 \right) \tag{3.2.1}$$

where

- F_t denotes the sigma algebra induced by Brownian motion up to time t,
- E denotes expectation with respect to the spot Libor measure.

We price a Bermudan style swaption using a Monte Carlo technique, which is based on the approach proposed by Longstaff and Schwartz towards American style pricing using simulation. In particular, at every exercise time, we must solve a linear least squares problem, and then decide whether to exercise the option.

Let ω denote a Libor rate sample path. At an exercise time, t', let

- $e(\omega)$ denote the sample value for the associated European style payoff, of the form (3.2.1),
- $h(\omega)$ be the sample holding value for the Bermudan style option.

We then solve in a least squares sense,

$$\sum_{j=1}^{5} A_j \left(e(\omega) \Big/ \frac{\sum_{i=1}^{n} e(\omega_i)}{n} \right)^{j-1} - h(\omega) = 0\,,$$

for the unknowns, $A_1,..., A_5$, where

- n is the total number of Monte Carlo sample paths,
- ω ranges over all sample paths such that the $e(\omega) > 0$ (i.e., the European style payoff sample value is positive).

After calculating the coefficients, $\{A_i\}$, the option is exercised at t' for the sample path, ω, if

$$e(\omega) > \sum_{j=1}^{5} A_j \left(e(\omega) \Big/ \frac{\sum_{i=1}^{n} e(\omega_i)}{n} \right)^{j-1}.$$

With respect to the implementation above for Bermudan style pricing, we note the following. From Formula (3.2.1) observe that, at exercise time t', the payoff includes terms of the form

$$N_{t'}\, E\left(\frac{r}{N_\tau} \middle| F_{t'} \right) \qquad (3.3.1)$$

where r is a τ -measurable interest rate quantity. To evaluate the European style payoff (3.3.1), we must calculate the sample value

$$\left[N_{t'}\, E\left(\frac{r}{N_\tau} \middle| F_{t'} \right) \right](\omega). \qquad (3.3.2)$$

Here we approximate (3.3.2) by

$$\left(r\, P(t',\tau) \right)(\omega) \qquad (3.3.3)$$

where $P(t',\tau)$ is the price at time t' of a zero-coupon bond with maturity, τ , and face value, \$1. In the above trades off accuracy for efficiency in calculation speed; we investigate, numerically, the accuracy of the approximation above with respect to European swaption pricing in Section 5.

Furthermore, the choice and number of basis functions to employ in the linear least squares problem above is open to experimentation.